QUANTUM UNIQUE ERGODICITY FOR MAPS ON $\mathbb{T}^2$

A thesis submitted in partial fulfilments for M.sc. degree of Tel Aviv University by

Lior Rosenzweig

September 2004

1

Acknowledgments

I would like to thank my supervisor Professor Zeev Rudnick, for his help in my research. The work with you has enlightened my eyes more than once.

I would also like to thank my parents for all their support, and to my loving wife Jelena.

Contents

Abstract

When a map is classically uniquely ergodic, it is expected that its quantization will posses quantum unique ergodicity. In this paper we give examples of Quantum Unique Ergodicity for the Kronecker map, the perturbed Kronecker map, and an upper bound for the rate of convergence. The case of degeneracies in the eigenspaces is also discussed. When the dimension of the eigenspace is greater then 1, one looks for symmetries of the quantized map called "Hecke Operators", and find a joint eigenbasis. We find here that for the mentioned maps and also for the irrational skew translation, this reduces the multiplicity to 1.

1   Introduction

1.1   background

One of the problems in Quantum Chaos is the asymptotic behavior of the expectation value in eigenstates. When quantizing classical dynamics on a phase space one constructs a Hilbert space of states ,$\mathcal{H}_h$, and an algebra of operators , the algebra of "quantum observables", that assigns for each smooth function on the phase space f an operator $\mathrm{Op}_h(f)$ where h implies dependence on Planck's constant h, and the dynamics is quantized to a unitary time evolution operator, U_h on $\mathcal{H}_h$. For any orthonormal basis of eigenfunctions of U_h, $\{\psi_j\}$, the expectation value of $\mathrm{Op}_h(f)$ in the eigenstate ψ_j is given by $\langle \mathrm{Op}_h(f)\psi_j, \psi_j \rangle$. The semiclassical limit of these is the limit where $h \to 0$. When the classical dynamics of a system is ergodic, it is known that the time average of the trajectories of the system converges to the space average. An analogue of this is given by Schnirelman's Theorem [10], which states that for ergodic system the expectation values of $\mathrm{Op}(f)$ converges to the phase space average of f, for all but possibly a zero density subsequence of eigenfunctions. This is referred to as quantum ergodicity. The case where there are no exceptional subsequences is referred to as quantum unique ergodicity (QUE).

A first example of QUE was given on the 2-torus $\mathbb{T}^2$, by Marklof and Rudnick [8], where the classical dynamics is an irrational skew translation, that is classically uniquely ergodic. In this paper we will give a family of more examples of QUE on the 2-torus, all of them are also classically uniquely ergodic.

4

When the phase space is $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ it is required that each state will be periodic in both position and momentum and thus Planck's constant is restricted to be an inverse of an integer $h = \frac{1}{N}$, and the Hilbert space is of dimension N, namely $L^2(\mathbb{Z}/N\mathbb{Z})$. The semiclassical limit in this case is the limit where $N \to \infty$. With this as the Hilbert space of state, the algebra of quantum observables attached to smooth functions on $\mathbb{T}^2$, $\text{Op}_N(f)$, are $N \times N$ matrices. Given a map A on $\mathbb{T}^2$, we define its quantization as a sequence of unitary operators on $L^2(\mathbb{Z}/N\mathbb{Z})$, $U_N(A)$ satisfying

$$\|U_N(A)^{-1}\text{Op}_N(f)U_N(A) - \text{Op}_N(f \circ A)\| \longrightarrow 0 \qquad \text{as } N \to \infty \qquad (1)$$

for all $f \in C^\infty(\mathbb{T}^2)$, where $f \circ A(p,q) = f(A(p,q))$. This is an analogue of Egorov's Theorem, and the eigenfunctions of $U_N(A)$ are analogues of eigenmodes. A famous example of a quantization of a map is of linear automorphism of $\mathbb{T}^2$ called the "CAT map",([5],[2]), that is if $A \in SL(2,\mathbb{Z})$. If $|trA| > 2$ that is if A is hyperbolic, and the map is known to be ergodic.

1.2 QUE for maps on the torus

The maps in this paper will be the Kronecker map ,the perturbed Kronecker map and skew translation.
The Kronecker map is

$$\begin{aligned} \tau_\alpha: \quad \mathbb{T}^2 &\to \quad \mathbb{T}^2 \\ x &\mapsto \quad x + \alpha \mod 1 \end{aligned}$$

where $\alpha = (\alpha_1, \alpha_2)$.
The perturbed Kronecker map is where one of the variables is perturbed with a nonlinear zero-averaged smooth function that is

$$\begin{aligned} \Phi_V^\alpha: \quad \mathbb{T}^2 &\to \quad \mathbb{T}^2 \\ \Phi_V^\alpha: \quad \begin{pmatrix} p \\ q \end{pmatrix} &\mapsto \begin{pmatrix} p + \alpha_1 \\ q + \alpha_2 + V(p) \end{pmatrix} \mod 1 \end{aligned}$$

The skew translation is

$$A_\alpha: \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} p + \alpha \\ q + 2p \end{pmatrix} \mod 1 \qquad (2)$$

where α is irrational number. The Kronecker map is known to be uniquely ergodic when $1, \alpha_1, \alpha_2$ are linearly independent over $\mathbb{Q}$. We will construct a

quantization of it by approximating α with rational numbers $\frac{a}{N} = \frac{(a_1, a_2)}{N}$. For rational numbers we have an exact Egorov, that is

$$U_{a,N}^{-1} \operatorname{Op}_N(f) U_{a,N} = \operatorname{Op}_N(f \circ \tau_{a/N})$$

and thus by the convergence of $\frac{a}{N}$ to α we will get (1). For this map we have the following theorem for polynomials:

Theorem 1.1. *Suppose $1, \alpha_1, \alpha_2$ are linearly independent over $\mathbb{Q}$. Let $f \in C^\infty(\mathbb{T}^2)$ be a polynomial. Then for all eigenfunctions ψ of $U_N(\tau_\alpha)$ we have that for N sufficiently large*

$$\langle \operatorname{Op}_N(f)\psi, \psi \rangle = \int_{\mathbb{T}^2} f(p,q) dpdq$$

For the more general case of smooth functions one needs to assume a certain restriction on α. we assume that α satisfy a certain diophantine inequality, that is there exists $\gamma > 0$ such that for all $n_1, n_2, k \in \mathbb{Z}$

$$|n_1\alpha_1 + n_2\alpha_2 + k| \gg \|(n_1, n_2)\|^{-\gamma} \tag{3}$$

This reduces the set of numbers rather than being all α such that $1, \alpha_1, \alpha_2$ are linearly independent over $\mathbb{Q}$ to a set of almost all α in Lebesgue measure sense. For these α we have,

Theorem 1.2. *Suppose $1, \alpha_1, \alpha_2$ are linearly independent over $\mathbb{Q}$ and satisfy (3) then for all eigenfunctions ψ of $U_N(\tau_\alpha)$*

$$|\langle \operatorname{Op}_N(f)\psi, \psi \rangle - \int_{\mathbb{T}^2} f(p,q) dpdq| \ll N^{-\theta} \quad \forall \theta > 0$$

The perturbed map is also uniquely ergodic. In fact we show that it is conjugate to τ_α and we also have QUE for it. We give an upper bound for the rate of convergence:

Theorem 1.3. *Suppose $1, \alpha_1, \alpha_2$ are linearly independent over $\mathbb{Q}$ and satisfy (3) then for all eigenfunctions ψ of $U_N(\Phi_V^\alpha)$*

$$|\langle \operatorname{Op}_N(f)\psi, \psi \rangle - \int_{\mathbb{T}^2} f(p,q) dpdq| \ll N^{-2}$$

These results on the asymptotic behavior for these maps is much faster comparing to other expected and known rates. For the skew translation (a proof for unique ergodicity can be found in [3]) it was proven by J. Marklof and Z. Rudnick in [8] that for generic α the exponent is bounded by $\frac{1}{4} + \epsilon$, and they also constructed special cases where the rate of convergence is arbitrarily slow. Another example is the CAT map where it is conjectured that the exponent of N is $\frac{1}{2} + \epsilon$.

1.3   Hecke operators

The degeneracies of the eigenspaces are also discussed. In spherical harmonics, when observing the eigenfunctions of the laplacian operator Δ on the 2-sphere S^2, it is known that the elements in the group of rotations of $\mathbb{R}^3$,$SO(3)$, commute with Δ and therefore act on its eigenspaces. A known phenomena is that if we fix a north pole $o \in S^2$, then there is a unique basis of joint eigenfunction of Δ, and of the commutative group of all rotations about o. Moreover it is known that $SO(3)$ acts irreducibly on the eigenspaces of Δ. A similar situation we find here. By constructing explicit eigenfunctions for the Kronecker map, we finds that the the eigenvalues of $U_N(A)$ are exactly all Dth roots of unity where $D = \gcd(a, N)$, $(\frac{a}{N} = \frac{(a_1, a_2)}{N}$ is the approximation of α), up to a constant rotation, and the dimensions of all eigenspaces is also D. Due to this fact we find that when $D > 1$ there are degeneracies in the eigenspaces, which cause a slower rate of convergence. In this case we find symmetries connected to $U_N(A)$. These are operators that commute with $U_N(A)$ called Hecke operators. This type of operators and eigenfunction were first introduced by Z. Rudnick and P. Kurlberg in [6] and were they showed that for the CAT map and for joint eigenfunctions of all Hecke operators, named Hecke eigenfunctions, the expectation value in these states converges to the classical phase-space average of the observable, (a phenomena that does not necessarily hold for general set of eigenfunctions [4])

We will find these Hecke operators for τ_α and for the skew translation. For τ_α we will find that these operators act irreducibly on the eigenspaces of $U_N(A)$, and for a choice of a generator of this group there is a unique basis of eigenfunctions, that is there are no joint degeneracies. The case of Φ_V^α is the same as of τ_α because of the fact that Φ_V^α is conjugate to τ_α. For the skew translation, we also find such Hecke operators , and see that there is a unique basis of Hecke eigenfunctions, for them the rate of convergence of

$\langle \text{Op}_N(f)\psi, \psi \rangle$ to the phase space average of f is always N^{-2}.

2   Background

We begin with a quantization procedure for maps on the 2-torus $\mathbb{T}^2$. The procedure can be find in full description in [6],[1]. We construct a Hilbert space of state $\mathcal{H}_h$ with respect to Planck's constant h, quantum observables, and a quantization of our maps.

2.1   Notations

We abbreviate $e(x) = e^{2\pi i x}$, and $e_N(x) = e(\frac{x}{N})$.

2.2   Hilbert space of state

Our classical phase space is $\mathbb{T}^2$. The elements of the Hilbert space are thus, distribution on the line $\mathbb{R}$ that are periodic in both position and momentum. Using the momentum representation of a wave-function ψ by the Fourier transform

$$\mathcal{F}_h\psi(p) = \frac{1}{\sqrt{h}} \int_{-\infty}^{\infty} \psi(q)e(\frac{-qp}{h})dq$$

we find that the requirements

$$\psi(q+1) = \psi(q) \quad \mathcal{F}_h\psi(p) = \mathcal{F}_h\psi(p+1)$$

restricts planck's constant h to be an inverse of integer $h = \frac{1}{N}$, and $\mathcal{H}_h$ consists of periodic point-mass distributions at the coordinates $Q = \frac{q}{N}$. We therefore find that the Hilbert space is of dimension N, and therefore denote $\mathcal{H}_N$, and we may identify it with $L^2(\mathbb{Z}/N\mathbb{Z})$, with the inner product

$$\langle \psi, \phi \rangle = \frac{1}{N} \sum_{Q \mod N} \psi(Q)\bar{\phi}(Q)$$

The Fourier transform is given by

$$\hat{\psi}(P) = [\mathcal{F}_N\psi](P) = \frac{1}{\sqrt{N}} \sum_{Q \mod N} \psi(Q)e_N(-QP)$$

8

and its inverse formula is

$$\psi(Q) = \left[\mathcal{F}_N^{-1}\hat{\psi}\right](Q) = \frac{1}{\sqrt{N}} \sum_{P \bmod N} \hat{\psi}(P)e_N(PQ)$$

2.3 Quantum observables

We now assign each classical observable, a smooth function $f \in C^\infty(\mathbb{T}^2)$, a quantum observable, that is an operator $\mathrm{Op}_N(f)$ on $\mathcal{H}_N$ that satisfy,

1. $\mathrm{Op}_N(\bar{f}) = \mathrm{Op}_N(f)^*$

2. $\mathrm{Op}_N(f)\,\mathrm{Op}_N(g) \sim \mathrm{Op}_N(fg) \qquad$ as $N \to \infty$

3. $\frac{1}{2\pi i N}\left[\mathrm{Op}_N(f), \mathrm{Op}_N(g)\right] \sim \mathrm{Op}_N(\{f,g\}) \qquad$ as $N \to \infty$

where $[A, B] = AB - BA$ is the commutator, and $\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial g}{\partial p}\frac{\partial p}{\partial q}$ are the Poisson bracket. The norm used is the induced norm from the inner product on $\mathcal{H}_N$.

A central role play the translation operators

$$[\mathrm{t}_1\psi](Q) = \psi(Q+1)$$

and

$$[\mathrm{t}_2\psi](Q) = e_N(Q)\psi(Q)$$

that are analogues of the of the differentiation and multiplication (respectively) operators. The Heisenberg's commutation relations are

$$\mathrm{t}_1^a\mathrm{t}_2^b = \mathrm{t}_2^b\mathrm{t}_1^a e_N(ab) \qquad \forall a, b \in \mathbb{Z}$$

Notice that

$$\mathcal{F}_N\mathrm{t}_1\mathcal{F}_N = \mathrm{t}_2$$

and

$$\mathcal{F}_N\mathrm{t}_2\mathcal{F}_N = \mathrm{t}_1^{-1}$$

With these operators we construct

$$T_N(n) = e_N(\frac{n_1 n_2}{2})t_2^{n_2}t_1^{n_1}, n = (n_1, n_2) \in \mathbb{Z}^2$$

whose action on a wave-function $\psi \in \mathcal{H}_N$ is

$$T_N(n)\psi(Q) = e^{\frac{i\pi n_1 n_2}{N}}e_N(n_2 Q)\psi(Q+n_1)$$

9

Notice that

$$T_N(n)^* = T_N(-n)$$

$$T_N(m)T_N(n) = e_N\left(\frac{\omega(m,n)}{2}\right)T_N(m+n) \tag{4}$$

where, $\omega(m,n) = m_1 n_2 - m_2 n_1$, and that T_N is a unitary operator. Finally for a general smooth function

$$f(x) = \sum_{n \in \mathbb{Z}^2} \hat{f}(n)e(n \cdot x)$$

where $x = (p,q)$. we define its quantization $\mathrm{Op}_N(f)$

$$\mathrm{Op}_N(f) = \sum_{n \in Z^2} \hat{f}(n)T_N(n) \tag{5}$$

and the conditions mentioned are all satisfied.

3 Quantization of maps and rate of convergence

When quantizing a map, we look for a sequence of unitary operators $U_N(A)$ on $\mathcal{H}_N$, the *quantum propagator*, whose iterates give the evolution of the quantum system, and that in the semiclassical limit, (the limit as $N \to \infty$ or $h \to \infty$), the quantum evolution follows the classical evolution as described in the following definition.

Definition 3.1 ("Egorov's Theorem"). *A quantization of a map $A : \mathbb{T}^2 \to \mathbb{T}^2$ is a sequence of unitary operators ,$\{U_N\}$, satisfying the following:*

$$\|U_N^{-1}\mathrm{Op}_N(f)U_N - \mathrm{Op}_N(f \circ A)\| \to 0 \qquad as \ N \to \infty \tag{6}$$

The stationary states of the quantum system are given by the eigenfunctions ψ of $U_N(A)$. We will find that for the maps studied in this paper the limiting expectation value of observables in normalized eigenstates converges (for all N) to the classical average of the observable, that is

$$\langle \mathrm{Op}_N(f)\psi, \psi \rangle \to \int_{\mathbb{T}^2} f \qquad as \ N \to \infty$$

3.1 Quantizing Kronecker map

In this section we construct a quantization to the Kronecker map.

Lemma 3.1. *suppose $\frac{(a_1, a_2)}{N}$ is a sequence of rational numbers such that $\frac{(a_1, a_2)}{N} = \frac{\vec{a}}{N} \xrightarrow[N \to \infty]{} \vec{\alpha}$ then the sequence $U_N := T_N(-a_2, a_1)$ is a quantization of Kronecker's map.*

Proof. First assume $f(x) = e_n(z) := e(n \cdot z)$. In this case we get that $\hat{f}(n) = 1, \hat{f}(m) = 0$ for $m \neq n$,and therefore $\mathrm{Op}_N(f) = T_N(n)$.
Denote $\tilde{a} := (-a_2, a_1)$,and notice that $n \cdot a = \omega(n, \tilde{a})$. Now

$$U_N^{-1} T_N U_N = T_N(-\tilde{a}) T_N(n) T_N(\tilde{a})$$

which due to (4), linearity and antisymmetry of $\omega(m, n)$ is

$$e_N(\omega(n, \tilde{a})) T_N(n) = e_N(n \cdot a)) T_N(n) \tag{7}$$

on the other hand,we have

$$(e_n \circ \tau_\alpha)(x) = e(n_1(p + \alpha_1) + n_2(q + \alpha_2)) = e(n \cdot \vec{\alpha}) e_n(x)$$

and so

$$\mathrm{Op}_N(e_n \circ \tau_\alpha) = e(n \cdot \vec{\alpha}) T_N(n) \tag{8}$$

And from (7),(8) we get that

$$\|U_N^{-1} T_N(n) U_N - e(n \cdot \alpha) T_N(n)\| = |e_N(n \cdot \vec{a}) - e_N(n \cdot \vec{\alpha})| \cdot \|T_N(n)\| \tag{9}$$

For a general function

$$f(x) = \sum_{n \in \mathbb{Z}^2} \hat{f}(n) e_n(x)$$

we have by (9), that

$$\|U_N^{-1} \mathrm{Op}_N(f) U_N - \mathrm{Op}_N(f \circ A)\| = \|U_N^{-1}\{\sum_{n \in \mathbb{Z}^2} \hat{f}(n) T_N(n)\} U_N - \sum_{n \in \mathbb{Z}^2} \hat{f}(n) e(n \cdot \alpha) T_N(n)\| =$$

$$\|\sum_{n \in \mathbb{Z}^2} \hat{f}(n)\{e_N(n \cdot a) - e(n \cdot \alpha)\} T_N(n)\| \leq \sum_{n \in \mathbb{Z}^2} |\hat{f}(n)| \cdot |e(n \cdot a) - e(n \cdot \alpha)| \cdot \|T_N(n)\|$$

And since T_N is a unitary operator so $\|T_N(n)\| = 1$,

$$|e_N(n \cdot \frac{\vec{a}}{N}) - e_N(n \cdot \vec{\alpha})| \ll \|n\| |\vec{\alpha} - \frac{\vec{a}}{N}|$$

11

we get

$$\|U_N^{-1}\operatorname{Op}_N(f)U_N - \operatorname{Op}_N(f \circ A)\| = |\vec{\alpha} - \frac{\vec{a}}{N}| \left(\sum_{n \in \mathbb{Z}^2} \|n\| \|\hat{f}(n)\| \right) = O(|\vec{\alpha} - \frac{\vec{a}}{N}|)$$

which goes to zero since $|\vec{\alpha} - \frac{\vec{a}}{N}| \to 0$ as $N \to \infty$ implying that U_N is a quantization of τ_α.

Remark 3.1. *Notice that for each N, we have exact Egorov for $\tau_{a/N}$, that is*

$$U_N^{-1}\operatorname{Op}_N(f)U_N = \operatorname{Op}_N(f \circ \tau a/N)$$

$\square$

3.2 Convergence of eigenstates

We now wish to give an upper bound for the remainder of

$$|\langle \operatorname{Op}_N(f)\psi, \psi \rangle - \int_{T^2} f| \tag{10}$$

where ψ is an eigenfunction of U_N. Actually we will prove the following:

Theorem 3.2. *Suppose $1, \alpha_1, \alpha_2$ are linearly independent over $\mathbb{Q}$. Then For any eigenfunction $\psi(Q)$ of U_N*

1. *If f is a polynomial then for N large enough,*

$$\langle \operatorname{Op}_N(f)\psi, \psi \rangle = \int_{T^2} f$$

2. *If α_1, α_2 are algebraic over $\mathbb{Q}$, then for $f \in C^\infty(\mathbb{T}^2)$*

$$\langle \operatorname{Op}_N(f)\psi, \psi \rangle - \int_{T^2} f = O(\frac{1}{N^\theta}) \ , \forall \theta > 0$$

3. *For almost every pair $\vec{\alpha} = (\alpha_1, \alpha_2)$ (in Lebesgue measure sense)*

$$\langle \operatorname{Op}_N(f)\psi, \psi \rangle - \int_{T^2} f = O(\frac{1}{N^\theta}) \ , \forall \theta > 0$$

12

To prove this Theorem we will start with the following lemma:

Lemma 3.3. *Denote by $\psi(Q)$ to be an eigenfunctions of U_N.*

1.
$$\langle \mathrm{Op}_N(f)\psi, \psi \rangle = \langle \mathrm{Op}_N(f^T)\psi, \psi \rangle \tag{11}$$

   where
$$f^T = \frac{1}{T} \sum_{t=0}^{T-1} f \circ \tau_{(a/N)}^t$$

2. *For $f(x) = e_n(x)$, $\langle T_N(n)\psi, \psi \rangle$ is identically zero for large enough N.*

Proof.   1. Since ψ is an eigenfunction of U_N then $U_N\psi = e(\phi)\psi$, and therefore for all t

$$\langle \mathrm{Op}_N(f)U_N^t\psi, U_N^t\psi \rangle = \langle e(t\phi)\,\mathrm{Op}_N(f)\psi, e(t\phi)\psi \rangle = \langle \mathrm{Op}_N(f)\psi, \psi \rangle$$

Now, since U_N is unitary we have

$$\langle \mathrm{Op}_N(f)U_N^t\psi, U_N^t\psi \rangle = \langle U_N^{-t}\,\mathrm{Op}_N(f)U_N^t\psi, \psi \rangle$$

and since
$$U_N^{-t}\,\mathrm{Op}_N(f)U_N^t = \mathrm{Op}_N(f \circ \tau_{a/N}^t)$$

we have (11).

2. fix $\vec{n} = (n_1, n_2) \in \mathbb{Z}^2$, $f(x) = e_n(x)$ and therefore $\mathrm{Op}_N(f) = T_N(n)$. Notice that for $f = e_n$ we have,

$$f^T = \frac{1}{T} \sum_{t=0}^{T-1} e_n \circ \tau_{(a/N)}^t = \frac{1}{T} \sum_{t=0}^{T-1} e(n_1(p + ta_1/N) + n_2(q + ta_2/N)) =$$

$$\frac{1}{T} e_n(p, q) \sum_{t=0}^{T-1} e_N((n_1 a_1 + n_2 a_2)t)$$

and for $T = N$ we have,

$$f^N = \begin{cases} f & \text{if } n_2 a_2 + n_1 a_1 = 0(\mod N) \\ 0 & \text{else} \end{cases} \tag{12}$$

13

and therefore,

$$\mathrm{Op}_N(f^N) = \begin{cases} \mathrm{Op}_N(f) & \text{if } n_2 a_2 + n_1 a_1 = 0(\mod N) \\ 0 & \text{else} \end{cases} \quad (13)$$

but

$$n_2 a_2 + n_1 a_1 = Nk \iff n_2 \frac{a_2}{N} + n_1 \frac{a_1}{N} = k \ \in \ \mathbb{Z}$$

$$\iff n_2\{\alpha_2 + \mathrm{O}(|\vec{\alpha} - \frac{\vec{a}}{N}|)\} + n_1\{\alpha_1 + \mathrm{O}(|\vec{\alpha} - \frac{\vec{a}}{N}|)\} = k \ \in \ \mathbb{Z}$$

and so we get

$$n_2\alpha_2 + n_1\alpha_1 + \mathrm{O}(\|n\||\vec{\alpha} - \frac{\vec{a}}{N}|)\} = k \in \mathbb{Z} \quad (14)$$

$1, \alpha_1, \alpha_2$ are linearly independent over $\mathbb{Q}$ so we can denote $0 < \delta = dist(n_1\alpha_1 + n_2\alpha_2, \mathbb{Z})$. Now assume that there exists infinitely many pairs $\vec{a} = (a_1, a_2)$ such that (10) is nonzero i.e. $n_2 a_2 + n_1 a_1 = Nk_{\vec{a}}$ .From (14) we get that

$$\mathrm{O}(\|n\||\vec{\alpha} - \frac{\vec{a}}{N}|) = |k + n_2\alpha_2 + n_1\alpha_1| \geqslant \delta > 0, N \to \infty \quad (15)$$

now since n is fixed and $|\vec{\alpha} - \frac{\vec{a}}{N}| \to 0$ as $N \to \infty$ we get a contradiction! so we can deduce that for $N \gg \|n\|$

$$|\langle \mathrm{Op}_N(f)\psi, \psi\rangle - \int_{\mathbb{T}^2} f|^2 = |\langle T_N(n)\psi, \psi\rangle| = 0$$

$\square$

Corollary 3.4. *For any eigenfunction ψ of U_N,*

1. *if f is a trigonometric polynomial, $\langle \mathrm{Op}_N(f)\psi, \psi\rangle$ is identically zero for large enough N.*

2. *For any $f \in C^\infty(\mathbb{T}^2)$,*

$$|\langle \mathrm{Op}_N(f)\psi, \psi\rangle - \int_{T^2} f| \to 0 \qquad as \ N \to \infty$$

14

Proof.   1. From the previous lemma we get that every trigonometric function has N such that (10) is identically zero so for a finite linear combination

$$\sum_{n=1}^{m} a_n e(n \cdot x)$$

simply choose the largest N given from $e_n(x), n = 1, \ldots, m$

2. For a general $f \in C^\infty(\mathbb{T}^2)$, we have

$$\mathrm{Op}_N(f) = \sum_{n \in \mathbb{Z}^2} \hat{f}(n) T_N(n)$$

For $\epsilon > o$, there exists R_0, such that $\forall R > R_0$,

$$\sum_{\|n\| > R} |\hat{f}(n)| < \epsilon$$

For the polynomial

$$P_R = \sum_{\|n\| < R} \hat{f}(n) e(n \cdot x)$$

there exists N_0, such that for all $N > N_0$

$$\langle \mathrm{Op}_N(P_R)\psi, \psi \rangle = 0$$

and so we have ,

$$|\langle \mathrm{Op}_N(f)\psi, \psi \rangle| \leqslant$$
$$|\langle \mathrm{Op}_N(P_R)\psi, \psi \rangle| + |\sum_{\|n\| > R} \hat{f}(n) \langle T_N(n)\psi, \psi \rangle| \leqslant \epsilon$$

for $N > N_0$.

$\square$

To finish the study of the upper bound for a general function we need to study the size of $n_1\alpha_1 + n_2\alpha_2 + k$ for $n_1, n_2, k \in \mathbb{Z}$, and assume that α satisfies a certain diophantine inequality that is $|n_1\alpha_1 + n_2\alpha_2 + k| \gg \frac{c(\alpha)}{\|n\|^\gamma}$ for some γ. Numbers like this are called diophantine.

Definition 3.2. *An l-tuple of real numbers $(\alpha_1, \ldots, \alpha_l)$ is called diophantine if they satisfy that there exists γ such that for any integers $(n_1, \ldots, n_l) \neq \vec{0}, k$*

$$|n_1\alpha_1 + \cdots + n_l\alpha_l + k| \gg \frac{c(\alpha)}{\|n\|^\gamma}$$

with this we have the following.

Corollary 3.5. *Suppose α is diophantine and that $|\vec{\alpha} - \frac{\vec{a}}{N}| \ll \frac{1}{N^{1-\epsilon}}$ then we have an upper bound for $|\langle \mathrm{Op}_N(f)\psi, \psi\rangle - \int_{\mathbb{T}^2} f| \ll \frac{1}{N^\theta}$ for any $\theta > 0$.*

Proof. A general function is of the following form

$$f(x) = \sum_{n \in \mathbb{Z}^2} \hat{f}(n)e_n(x)$$

without loss of generality we can assume that $\int_{\mathbb{T}^2} f = 0$ and so divide $\mathrm{Op}_N(f)$ into two sums: $\mathrm{Op}_N(f) = \sum_{n \in \mathbb{Z}^2} \hat{f}(n)T_N(n) = I_1 + I_2$ where $I_1 = \sum_{\|n\| \leqslant R} \hat{f}(n)T_N(n), I_2 = \sum_{\|n\| > R} \hat{f}(n)T_N(n)$. Now as seen earlier, the case when $|\langle T_N(\vec{n})\psi, \psi\rangle| \neq 0$ can only happen when

$$\mathrm{O}(\frac{\|n\|}{N^{1-\epsilon}}) = |k + n_2\alpha_2 + n_1\alpha_1|$$

but our assumption is that there exists γ such that for all integer coefficients $k + n_2\alpha_2 + n_1\alpha_1 \gg \frac{1}{\|n\|^\gamma} \gg \frac{1}{R^\gamma}$ and so define $N^{1-\epsilon} = R^{1+\gamma+\delta}$ for some $\delta > 0$ and we get that

$$\frac{R}{N^{1-\epsilon}} \geqslant \frac{\|n\|}{N^{1-\epsilon}} \gg k + n_2\alpha_2 + n_1\alpha_1 \gg \frac{1}{\|n\|^\gamma} \gg \frac{1}{R^\gamma}$$

and for $N^{1-\epsilon} = R^{1+\gamma+\delta}$ this gives a contradiction and so $I_1 = 0$ for large enough N. For I_2 we use the rapid decay of the Fourier coefficients:

$$|I_2| = |\sum_{\|n\| > R} \hat{f}(n)T_N(n)| \leqslant \sum_{\|n\| > R} \|\hat{f}(n)T_N(n)\| = \sum_{\|n\| > R} |\hat{f}(n)| \leqslant \frac{1}{R^b} = \frac{1}{N^\theta}$$

for any chosen θ. $\qquad\square$

For algebraic numbers we have this inequality by the following well known theorem, (a proof is given in appendix A.2):

Theorem 3.6. *Suppose $\vec{\alpha} = (\alpha_1, \ldots, \alpha_m)$ are linearly independent over $\mathbb{Q}$ then there exists $D = D(\alpha)$ such that*

$$|n_1\alpha_1 + n_m\alpha_m + k| \gg \frac{c(\vec{\alpha})}{\|n\|^{D-1}}$$

For the more general $\vec{\alpha}$ we need the following theorem by Khintchine [9]:

Theorem 3.7. *Almost no pair (α_1, α_2) is very well approximable that is that for almost any pair there exists $\delta = \delta(\alpha_1, \alpha_2)$ such that there are only finite many integers $m = (m_1, m_2), k$ such that the following inequality holds:$|m_1\alpha_1 + m_2\alpha_2 + k| \geqslant \frac{1}{\|m\|^{2+\delta}}$*

3.3 Perturbed Kronecker map

Another family of uniquely ergodic maps on $\mathbb{T}^2$, is the perturbed Kronecker map. we see in this section that it is uniquely ergodic, due to the fact that it is conjugate to the Kronecker map itself, and in the following section we form a quantization for it.

Define the following shear perturbation:

$$\Phi_V : \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} p \\ q + V(p) \end{pmatrix}$$

and the perturbed Kronecker map:

$$\Phi_V^\alpha : \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} p + \alpha_1 \\ q + \alpha_2 + V(p) \end{pmatrix}$$

where $V(p) \in C^\infty(\mathbb{T})$ satisfies $\int_0^1 V(p)dp = 0$. In order to prove the unique ergodicity of this map, we will use the following Lemma that shows that the perturbed map is conjugate to the Kronecker map.

Lemma 3.8. *Suppose α_1 is irrational.*

1. *If $V(p)$ is a polynomial we have that*

$$\tau_\alpha \circ \Phi_V = \Phi_h \circ \tau_\alpha \circ \Phi_h^{-1}$$

2. *If α_1 is diophantine then (1) holds for any $V \in C^\infty(\mathbb{T})$*

for some $h = h_V \in C^\infty(\mathbb{T})$

Proof. 1. The RHS of (1) is

$$\Phi_{h_k} \circ \tau_\alpha \circ \Phi_{h_k}^{-1}(p,q) = \begin{pmatrix} p + \alpha_1 \\ q + \alpha_2 + h_k(p + \alpha_1) - h_k(p) \end{pmatrix}$$

define $h_k(p) = \frac{e(kx)}{e(k\alpha_1)-1}$ (which is well defined for all k only if α_1 is irrational). $h_k(p)$ satisfy that $e(kp) = h_k(p + \alpha_1) - h_k(p)$ and therefore we get (1),and by linearity we get that(1) holds for every polynomial.

2. For $V \in C^\infty(\mathbb{T})$, α_1 diophantine , we have that $|e(k\alpha_1) - 1| \sim \{k\alpha\} \gg \frac{1}{|k|^\gamma}$ and we get that

$$\sum_{k\in\mathbb{Z}} |\hat{V}(k)h_k(p)| \ll \sum_{k\in\mathbb{Z}} |\hat{V}(k)||k|^\gamma$$

converges absolutely, and so define $h_V(p) = \sum_{k\in\mathbb{Z}} \hat{V}(k)h_k(p)$ and $h_V(p)$ satisfy $h_V(p + \alpha_1) - h_V(p) = V(p)$ since h_k satisfy that for every k and due to the absolute convergence of the series.

$\square$

with Φ_V^α described as a conjugate of τ_α we have the following result:

Theorem 3.9. *Suppose $1, \alpha_1, \alpha_2$ are linearly independent over $\mathbb{Q}$. Then,*

1. *For $V(p)$ a polynomial, Φ_V^α is uniquely ergodic.*

2. *For α diophantine and $V(p) \in C^\infty(\mathbb{T})$ then Φ_V^α is uniquely ergodic.*

Proof. The proof is a corollary of Lemma 3.8 and therefore the same for both parts.
We will first show that Lebesgue measure is Φ_V^α invariant. Suppose $f(p,q) \in L^1(\mathbb{T}^2)$. Then $f \circ \Phi_V^\alpha(p,q) = f(p, q + V(p))$ and so

$$\int_0^1 \int_0^1 f(p + \alpha_1, q + V(p) + \alpha_2)dqdp = \int_0^1 \int_0^1 f(p,q)dqdp$$

by standard change of variables. Now, assume μ is an invariant measure of Φ_V^α. since $\Phi_V^\alpha = \Phi_h \circ \tau_\alpha \circ \Phi_h^{-1}$ for some $h \in C^\infty(\mathbb{T})$, then $\Phi_h \circ \mu$ is invariant measure of τ_α, but there exists only one such measure and which is Lebesgue measure m, that is $\Phi_h \circ \mu = m$ is Lebesgue measure. Φ_h is an invertible map, that preserves Lebesgue measure, so $\mu = \Phi_h^{-1} \circ m = m$ therefore Φ_V^α is uniquely ergodic. $\square$

3.4 Quantizing the perturbed maps

In order to quantize the perturbed Kronecker map, in this section we prove an Egorov theorem for the quantization of the map:

$$\Phi_v : p \mapsto p, \qquad q \mapsto q + v(p) \mod 1$$

where $v(p) \in C^\infty(\mathbb{T})$, $\int_\mathbb{T} v = 0$.

Denote $U_v(N) = e^{iNV(\frac{Q}{N})}$, where $V'(p) = -v(p)$. The quantization was constructed by Marklof and O'Keefe, in [7].

Lemma 3.10. *For fixed* $n = (n_1, n_2)$ *we have*

$$||U^{-1} \mathrm{Op}_N(e_n)U - \mathrm{Op}_N(e_n \circ \Phi)|| \ll \frac{|n_1|^3}{N^2}$$

Proof. It suffices to show that

$$\langle \left(U_v^{-1} \mathrm{Op}_N(e_n)U_v - \mathrm{Op}_N(e_n \circ \Phi_v) \right) \psi, \psi \rangle \ll \frac{|n_1|^3}{N^2}$$

for all $\psi \in L^2(\mathbb{Z}/N\mathbb{Z})$ with $||\psi|| = 1$. So take the normalized delta-functions $u_Q(Q') := \sqrt{N}\delta_Q(Q')$ as the orthonormal basis of $L^2(\mathbb{Z}/N\mathbb{Z})$, and lets compute matrix elements, that is for $Q_1, Q_2 \mod N$ compute (we abbreviate $g_N(x) := e^{-iNV(x)}$):

$$\langle U_v^{-1}T_N(n)U_v u_{Q_1}, u_{Q_2}\rangle = \langle T_N(n)U_v u_{Q_1}, U_v u_{Q_2}\rangle$$

$$= e^{i\pi n_1 n_2/N} \sum_{Q' \mod N} g_N(\frac{Q' + n_1}{N})\delta_{Q_1}(Q' + n_1)e_N(n_2Q')\bar{g}_N(\frac{Q'}{N})\delta_{Q_2}(Q')$$

$$= e^{i\pi n_1 n_2/N}g_N(\frac{Q_2 + n_1}{N})\overline{g_N(\frac{Q_2}{N})}e_N(n_2Q)\delta_{Q_1}(Q_2 + n_1)$$

$$= e^{-iNV(\frac{(Q_2+n_1)}{N})}e^{iNV(2\pi \frac{kQ_2}{N})}e(n_2(\frac{Q_2}{N}) + \frac{n_1}{2N})\delta_N(Q_1 - Q_2, n_1)$$

where $\delta_N(a, b) = 1$ if $a = b \mod N$ and is zero otherwise. Now expand the exponents, keeping n_1, n_2 fixed while $N \to \infty$, using Taylor expansion of $V(p)$ around $x_0 = \frac{Q}{N} + \frac{n_1}{2N}$

$$-iNV(\frac{Q_2 + n_1}{N}) + iNV(\frac{Q_2}{N}) = in_1v(\frac{Q_2}{N} + \frac{n_1}{2N}) + O(\frac{|n_1|^3}{N^2})$$

19

to find that

$$\langle U_v^{-1} T_N(n) U_v u_{Q_1}, u_{Q_2} \rangle$$
$$= e^{i\pi n_1 n_2/N} e^{in_1 v(\frac{kQ_2}{N})} e(\frac{n_2 Q_2}{N}) \delta_N(Q_1 - Q_2, n_1) \left(1 + O(\frac{|n_1|^3{}^2}{N})\right) \quad (16)$$

To compute $\langle \mathrm{Op}_N(e_n \circ \Phi) u_{Q_1}, u_{Q_2} \rangle$, we will use an alternative representation of $\mathrm{Op}_N(f)$

$$[\mathrm{Op}_N(f)]\psi(Q) = \sum_{m \in \mathbb{Z}} a(m, \frac{Q}{N} + \frac{m}{2N})\psi(Q + m)$$

where

$$a(m, p) = \int_{\mathbb{T}} f(p, q) e(-mq) dq$$

In our case where $f(p, q) = e_n \circ \Phi(p, q)$, we have $a(n_1, p) = e(n_2 p + n_1 v(p))$ for $m = n_1$ and 0 otherwise. and then

$$\langle \mathrm{Op}_N(e_n \circ \Phi_v) u_{Q_1}, u_{Q_2} \rangle = \sum_{m \in \mathbb{Z}} \langle a(m, \frac{Q}{N} + \frac{m}{2N}) u_{Q_1 + n_1}, u_{Q_2} \rangle =$$
$$e(n_2(\frac{Q}{N} + \frac{m}{2N}) + n_1 v(2\pi(\frac{Q}{N} + \frac{m}{2N}))) \delta_N(Q_1 - Q_2, n_1) \quad (17)$$

and so comparing with (16) gives

$$\langle U_v^{-1} T_N(n) U_v u_{Q_1}, u_{Q_2} \rangle - \langle \mathrm{Op}_N(e_n \circ \Phi_v) u_{Q_1}, u_{Q_2} \rangle = O(\frac{|n_1|^3}{N^2} \delta_N(Q_1 - Q_2, n_1))$$

Now take a general normalized wavefunction $\psi \in L^2(\mathbb{Z}/N\mathbb{Z})$:

$$\psi = \sum_Q c(Q) u_Q, \quad c(Q) = \psi(Q)/\sqrt{N}, \quad \sum_{Q \mod N} |c(Q)|^2 = 1$$

Then

$$\langle (U_v^{-1} T_N(n) U_v - \mathrm{Op}_N(e_n \circ \Phi_v))\psi, \psi \rangle \ll \sum_{Q_1, Q_2 \mod N} c(Q_1)\overline{c(Q_2)} \frac{|n_1|^3}{N^2} \delta_N(Q_1 - Q_2, n_1)$$
$$= \frac{|n_1|^3}{N^2} \sum_{Q_2 \mod N} c(Q_2 + n_1)\overline{c(Q_2)}$$

20

and applying the Cauchy-Schwartz inequality, the sum is bounded by $\sum_{Q \mod N} |c(Q)|^2 = 1$. Thus we find that

$$\langle (U_v^{-1} T_N(n) U_v - \mathrm{Op}_N(e_n \circ \Phi)) \psi, \psi \rangle \ll \frac{|n_1|^3}{N^2}$$

$\square$

Theorem 3.11. *For every function $f \in C^\infty(\mathbb{T}^2)$ we have*

$$|\langle (U_v^{-1} \mathrm{Op}_N(f) U_v - \mathrm{Op}_N(f \circ \Phi_v)) \psi, \psi \rangle| \ll \frac{c(f)}{N^2} \tag{18}$$

Proof. A general function $f(p,q) = \sum_{n \in \mathbb{Z}^2} \hat{f}(n) e_n(p,q)$ and $\mathrm{Op}_N(f) = \sum_{n \in \mathbb{Z}^2} \hat{f}(n) T_N(n)$. Split the sum at $R \in \mathbb{N}$ and we have

$$|\sum_{\|n\| \leqslant R} \hat{f}(n) \langle (U_v^{-1} T_N(n) U_v - \mathrm{Op}_N(e_n \circ \Phi_v)) \psi, \psi \rangle| \quad \ll \quad \frac{\sum_{n \in \mathbb{Z}} |n_1|^3 \hat{f}(n)}{N^2}$$

For the remainder $\sum_{\|n\| > R} |\hat{f}(n) \langle (U_v^{-1} T_N(n) U_v - \mathrm{Op}_N(e_n \circ \Phi_v)) \psi, \psi \rangle|$ we have an upper bound of $\frac{1}{R^{10}}$ Now, since $\sum_{n \in \mathbb{Z}^2} \hat{f}(n)(e_n \circ \Phi_v) = f \circ \Phi_v$, we have

$$|\langle (U_v^{-1} \mathrm{Op}_N(f) U_v - \mathrm{Op}_N(f \circ \Phi_v)) \psi, \psi \rangle| \ll \frac{c(f)}{N^2} + R^{-10}$$

and for $R = N^{\frac{1}{2}}$ concludes the proof. $\square$

3.5 QUE for perturbed Kronecker map

In this section we will study the asymptotic behaviour of the matrix elements related to the perturbed Kronecker map. The main tool will be lemma 3.8 that connects the perturbed map to the unperturbed map.

Using the equality in Lemma (3.8) we can describe the quantization of $\Phi_V^\alpha = \tau_\alpha \circ \Phi_v$ as follows:

Theorem 3.12. *Denote $U_N = U_h(N)^{-1} U_\tau(N) U_h(N)$ where $U_\tau(N)$ is the quantization of τ_α, then we have*

$$\|U_N^{-1} \mathrm{Op}_N(f) U_N - \mathrm{Op}_N(f \circ \tau_\alpha \Phi_v)\| \ll N^{-1} \tag{19}$$

Proof. We already know that

$$\|U_h^{-1}\operatorname{Op}_N(f)U_h - \operatorname{Op}_N(f\circ\Phi_h)\| = O(N^{-2})$$

and that

$$\|U_\tau(N)^{-1}\operatorname{Op}_N(f)U_\tau - \operatorname{Op}_N(f\circ\tau)\| = O(N^{-1})$$

and thus using the equality in Lemma (3.8) we conclude the proof $\qquad\square$

Remark 3.2. *The set $\{\psi_j = U_h(N)^{-1}\psi_j^\tau\}$ form a basis of eigenfunctions of U_N, where $\{\psi_j^\tau\}$ is a basis of eigenfunctions for U_τ.*

With this representation of the eigenfunctions we can give an upper bound for the asymptotic behavior of the matrix elements:

Theorem 3.13. *For every $f \in C^\infty(\mathbb{T}^2)$, α diophantine we have:*

$$|\langle\operatorname{Op}_N(f)\psi_j,\psi_j\rangle - \int f| \ll N^{-2}$$

Proof. Without loss of generality we will assume that $\int f = 0$. By definition we have

$$\langle\operatorname{Op}_N(f)\psi_j,\psi_j\rangle = \langle\operatorname{Op}_N(f)U_h^{-1}\psi_j^\tau, U_h^{-1}\psi_j^\tau\rangle$$

and since U_h is unitary we have

$$\langle\operatorname{Op}_N(f)\psi_j,\psi_j\rangle = \langle U_h\operatorname{Op}_N(f)U_h^{-1}\psi_j^\tau, \psi_j^\tau\rangle$$

Now using Theorem 3.11 we get,

$$|\langle U_h\operatorname{Op}_N(f)U_h^{-1}\psi_j^\tau, \psi_j^\tau\rangle - \langle\operatorname{Op}_N(f\circ\Phi_h)\psi_j^\tau, \psi_j^\tau\rangle| \ll N^{-2}$$

since ψ_j is a normalized wavefunction , but using that $f\circ\Phi_h$ is still a $C^\infty(\mathbb{T}^2)$ we have that the second term is $O(N^{-10})$ and therefore

$$\langle\operatorname{Op}_N(f)\psi_j,\psi_j\rangle \ll N^{-2}$$

$\qquad\square$

Remark 3.3. *The upper bound found here is valid only for the quantization of described here which includes an arbitrary choice of a sequence that converges to α by rational numbers. since this quantization is not unique, and since the operators $\|U_N(a) - U_N(a')\| \sim \frac{1}{N}$ this upper bound only applies with the specific eigenfunctions for a specific chosen convergent sequence for α.*

4 Explicit eigenfunctions and eigenvalues

We now compute explicit eigenfunctions of U_N and find its eigenvalues. An eigenfunction of U_N should satisfy the following condition:

$$e_N(-\frac{a_1 a_2}{2})e_N(a_1 Q)\psi(Q - a_2) = e_N(\phi)\psi(Q) \tag{20}$$

Denote

$$D_2 = gcd(a_2, N) \quad M_2 = \frac{N}{D_2}$$

and for each $0 \le j \le N - 1$ match a pair (η, l) such that $j = \eta + lD_2 \quad \eta \in [1, D_2], \quad l \in [0, M_2 - 1]$.

Lemma 4.1. *The functions*

$$\psi_{\eta,l}^{(2)}(Q) = \begin{cases} \sqrt{D_2}e_N(\nu lD_2 + \frac{a_1 a_2}{2}\nu(\nu - M_2)) & P \equiv \eta - \nu a_2 (mod\ N) \\ 0 & P \not\equiv \eta (mod\ D_2) \end{cases}$$

are an orthonormal basis of eigenfunctions of U_N with eigenvalues

$$\phi_{\eta,l}^{(2)} = lD_2 + a_1\eta - \frac{a_1 a_2}{2}M_2$$

Proof. fix $\psi_{\eta,l}^{(2)}(\eta) = \sqrt{D_2}$ according to (20)we can deduce that

$$\psi(Q - a_2) = e_N(\frac{a_1 a_2}{2})e_N(-a_1 Q)e_N(\phi)\psi(Q)$$

and,

$$\psi(\eta - \nu a_2) = e_N(\nu\frac{a_1 a_2}{2} - a_1\sum_{m=0}^{\nu-1}(\eta - ma_2) + \nu\phi) \tag{21}$$

for $\nu = M_2$ we get $N|M_2 a_2$ and therefore $\psi_{\eta,l}^{(2)}(\eta - M_2 a_2) = \psi_{\eta,l}^{(2)}(\eta)$ or

$$e_N(M_2\frac{a_1 a_2}{2} - a_1\sum_{m=0}^{M_2-1}(\eta - ma_2) + M_2\phi_{\eta,l}^{(2)}) = 1$$

for $\psi_{\eta,l}^{(2)}$ consider

$$M_2\frac{a_1 a_2}{2} - a_1\sum_{m=0}^{M_2-1}(\eta - ma_2) + M_2\phi_{\eta,l}^{(2)} = lN \quad \Leftrightarrow$$

$$\phi_{\eta,l}^{(2)} = lD_2 - \frac{a_1 a_2}{2} + \eta a_1 - \frac{a_1 a_2}{2}(M_2 - 1) = lD_2 + \eta a_1 - \frac{a_1 a_2}{2}M_2$$

put $\phi_{\eta,l}^{(2)}$ in (21) and we get the required functions. $\qquad\square$

Proposition 4.2. *The multiplicity of an eigenphase ϕ is exactly $D = (a_1, a_2, N)$. The eigenphases are $kD - \frac{a_1 a_2}{2} M_2 (mod\ N)$.*

Proof. since $-\frac{a_1 a_2}{2} M_2$ is a constant depending only on a_1, a_2, N it suffice to show that the values achieved by $lD_2 + a_1 \eta (mod\ N)$ $l \in [0, M_2-1], \eta \in [1, D_2]$ are

1. of multiplicity D

2. are all multiples of D.

2. is true since $\forall\ l, \eta$ $D = (a_1, D_2)|lD_2 + a_1\eta$. suppose that $\phi \equiv kD$. there exist η unique modulo $\frac{D2}{D}$ (and therefore there are exactly D as such) such that $a_1\eta \equiv \phi(mod\ D_2)$. for such η there exists a unique $l(mod\ M_2)$ such that $lD_2 + a_1\eta \equiv \phi$ and so we have shown that ϕ is an eigenvalue of multiplicity D. $\qquad\square$

The multiplicity D of the eigenspaces may be to 1 if we choose a_2 to be coprime to N (and thus $D = 1$). This can be achieved with a rate of convergence of $\|\alpha - \frac{a}{N}\| \ll N^{1-\epsilon}$ which is the rate assumed for the rate of convergence we achieved in section 3.2 (see section A.1 for a proof of this rate). In this case we can write the eigenfunctions more explicitly:

Proposition 4.3. *suppose that the vectors $\frac{\vec{a}}{N}$ satisfy that $(a_2, N) = 1$, define $\psi_0(Q) = e_N(\frac{a_1 a_2^{-1} Q^2}{2})$, ($a_2^{-1}$ is the inverse of a_2 modulo N). Then:*

1. *ψ_0 is a well defined function of $\mathbb{Z}/N\mathbb{Z}$.*

2. *ψ_0 is an eigenfunction of U_N with eigenvalue 1.*

Proof.   1. since $(a_2, N) = 1$ then a_2^{-1} exists modulo N. For ψ_0 to be well defined it should give the same value for any residue class modulo N.Checking $\psi_0(Q + N)$ we get

$$\psi_0(Q + N) = e_N(ba_1 a_2^{-1}(Q + N)^2) \ = $$
$$e_N(\frac{a_1 a_2^{-1} Q^2}{2})e_N(\frac{a_1 a_2^{-1}(2N + N^2)}{2}) \ = \ e_N(\frac{a_1 a_2^{-1} Q^2}{2})e(\frac{a_1 a_2^{-1}(2 + N)}{2})$$

since a_1 is even we get that $e(\frac{a_1 a_2^{-1}(2+N)}{2}) = 1$

2. By definition we get that,

$$T_N(\tilde{a})\psi_0(Q) = e^{\frac{-i\pi a_1 a_2}{N}} e_N(a_1 Q)\psi_0(Q - a_2) \qquad (22)$$

Now ,

$$\psi_0(Q - a_2) = e_N(ba_1 a_2^{-1}(Q - a_2)^2) = e_N(ba_1 a_2^{-1}(Q^2 - 2a_2 + a_2^2))$$

which is

$$e_N(\frac{a_1 a_2^{-1} Q^2}{2})e_N(-a_1(a_2^{-1}a_2))e_N(\frac{a_1 a_2^{-1} a_2^2}{2})$$

by definition of a_2^{-1} we get that $a_2^{-1}2a_2 \equiv 1(mod\ N), a_1 a_2^{-1} a_2^2 \equiv a_1 a_2(mod\ N)$ and therefore,

$$T_N(\tilde{a})\psi_0(Q) = \psi_0(Q)$$

$\square$

Corollary 4.4. *Define* $\phi_c(Q) := e_N(cQ), c \in \mathbb{Z}/N\mathbb{Z}$ *the functions*

$$\psi_j := \psi_0(Q)\phi_j(Q), j = 1 \dots n$$

is a basis of eigenfunctions of U_N*, with eigenvalues* $e_N(-a_2 c), c = 1 \dots N$*.*

Proof. All ϕ_c are eigenfunctions of the operator

$$[T]\phi(Q) = \phi(Q - a_2)$$

and therefore multiplying each one with ψ_0 we get an eigenfunction of U_N with eigenvalue $e_N(-a_2 c)$ $\square$

5   Hecke operators

Another way of restricting the degeneracy of the eigenspaces is the Hecke operators of U_N. These are operators that commute with $U_N(A)$, and therefore act on its eigenspace. In the following section, we find the Hecke operators of the Kronecker map the perturbed Kronecker map and irrational skew translation.

5.1 Hecke operators for Kronecker map

Here we find the Hecke operators of $U_\tau(N)$, (and therefore of $U_v(N)$ because it is conjugate to $U_\tau(N)$), and see that they act irreducibly on the the eigenspaces of $U_\tau(N)$. Furthermore, we find that the group of Hecke operators , denoted by $\mathcal{G}$ is generated by 3 operators, and for each generator, we find a unique basis of joint eigenfunctions of all operators in the commutative subgroup of $\mathcal{G}$ containing $U_\tau(N)$ and the generator.

Definition 5.1. *Operators contained in a commutative subgroup of all operators that commute with operator T are called Hecke operators of T.*

Proposition 5.1. *1. The operators $T_N(\tilde{b}) = T_N((-b_2, b_1))$ such that*

$$a_1 b_2 - a_2 b_1 \equiv 0 (\mathrm{mod} N)$$

all commute with U_N

2. The solution for this equation is:

$$(x, y) = (a_2 + \frac{k_1 a_2}{(a_1, a_2)} + \frac{l_1 N}{(a_1, N)}, -a_1 - \frac{k_1 a_1}{(a_1, a_2)} + \frac{l_2 N}{(a_2, N)})$$

with $k_1, l_1, l_2 \in \mathbb{Z}$.

Proof. 1. recall (4) and consider the following:

$$T_N(\tilde{b}) T_N(\tilde{a}) = e_N(\frac{\omega(\tilde{b}, \tilde{a})}{2}) T_N(a + b)$$

$$T_N(\tilde{a}) T_N(\tilde{b}) = e_N(\frac{\omega(\tilde{a}, \tilde{b})}{2}) T_N(b + a)$$

and so in order for

$$T_N(\tilde{b}) T_N(\tilde{a}) = T_N(\tilde{a}) T_N(\tilde{b}) \tag{23}$$

to occur we must have that $\omega(\tilde{b}, \tilde{a}) \equiv \omega(\tilde{a}, \tilde{b})(\mod 2N)$ or

$$2\omega(\tilde{b}, \tilde{a}) = 0(\mod 2N) \leftrightarrow$$
$$\omega(\tilde{b}, \tilde{a}) = 0(\mod N)$$

that is $a_1 b_2 - a_2 b_1 \equiv 0(\mod N)$

2. This equation is equivalent to the diophantine equation

$$a_1 x - a_2 y + Nz = 0$$

in order to solve this we will first solve for $z = 0$ and we get a 2 variables linear equation which solution is $(x, y) = (a_2 + \frac{k_1 a_2}{(a_1, a_2)}, a_1 + \frac{k_1 a_1}{(a_1, a_2)})$. Now the general solution will be adding freely multiples of N to each variables i.e the solution is

$$(x, y) = (a_2 + \frac{k_1 a_2}{(a_1, a_2)} + \frac{l_1 N}{(a_1, N)}, -a_1 - \frac{k_1 a_1}{(a_1, a_2)} + \frac{l_2 N}{(a_2, N)}) =$$

$$(a_2, -a_1)(1 + \frac{k}{(a_1, a_2)}) + N(\frac{l_1}{(a_1, N)}, \frac{l_2}{(a_2, N)})$$

with $k_1, l_1, l_2 \in \mathbb{Z}$.

$\square$

Remark 5.1. *Notice that all these operators can be described as the group generated by* $T_N(\frac{\tilde{a}}{(a_1, a_2)}), t_2^{M_2}, t_1^{M_1}$ *that is*

$$\mathcal{G} = < T_N(\frac{\tilde{a}}{(a_1, a_2)}), t_2^{\frac{N}{(a_2, N)}}, t_1^{\frac{N}{(a_1, N)}} >$$

This group is commutative only if $(a_1, a_2, N) = 1$.

Although this group is not commutative, by choosing one of its generators described, we get a commutative subgroup of it, having unique Hecke eigenfunction. This is described in the following.

Proposition 5.2. *1. The functions*

$$\psi_{\eta, l}^{(1)}(P) = \begin{cases} \sqrt{D_1} e_N(\nu l D_1 + \frac{a_1 a_2}{2}\nu(\nu - M_2)) & P \equiv \eta - \nu a_2 (mod\ N) \\ 0 & P \not\equiv \eta (mod\ D_1) \end{cases}$$

are an orthonormal basis of eigenfunctions of $U_N^{\mathcal{F}_N}$ *with eigenvalues*

$$\phi_{\eta, l}^{(1)} = l D_1 + a_2 \eta + \frac{a_1 a_2}{2} M_1$$

2. Denote $g = (a_1, a_2)$, $D_2' = (\frac{a_2}{g}, N)$. *The functions*

$$\psi_{\eta, l}^{(2)}(Q) = \begin{cases} \sqrt{D_2'} e_N(\nu l D_2' + \frac{a_1 a_2}{2g^2}\nu(\nu - \frac{N}{D_2'})) & P \equiv \eta - \nu\frac{a_2}{g} (mod\ N) \\ 0 & P \not\equiv \eta (mod\ D_2') \end{cases}$$

The proof is the same as in (4.1) and the fact that

$$t_1^{\mathcal{F}_N} = t_2 \quad , t_2^{\mathcal{F}_N} = t_1^{-1}$$

Remark 5.2. *according to (4.2) we can deduce that $T_N(\frac{\tilde{a}}{(a_1,a_2)})$ has a unique basis of eigenfunctions since the multiplicity of all eigenvalues is 1.*

Proposition 5.3. *For the commutative subgroup of $\mathcal{G}$,*

$$\mathcal{G}_1 = < T_N(\tilde{a}), t_2^{M_2}, t_1^{\frac{N}{(a_1,a_2,N)}} >$$

the functions $\psi_{\eta,l}$ form a unique Hecke eigenfunctions basis

Proof. We can describe $\psi_{\eta,l}(Q)$ as

$$\psi_{\eta,l}(Q) = \sum_{\nu \mod D} c_{j,\nu} \delta_{\eta+\nu a}(Q)$$

The functions $\delta_{\eta+\nu a}$ are eigenfunctions of $t_2^{M_2}$ with eigenvalues $e_N(M\eta)$, therefore $\psi_{\eta,l}$ is an eigenvalue of $t_2^{M_2}$ with eigenvalue $e_N(M\eta)$ Now suppose $\psi_{\eta,l}, \psi_{\eta',l'}$ have the same eigenvalue for all $\mathcal{G}_1$ then

$$M\eta \equiv M\eta'(\mod N)$$

and therefore $\eta \equiv \eta'(\mod D)$, and from the definition of the eigenvalues of $T_N(\tilde{a})$ if η is fixed then l is determined modulo M and therefore $\psi_{\eta,l} = \psi_{\eta',l'}$. since the joint multiplicity of all functions for all elements of $\mathcal{G}_1$ is 1 then this is the unique basis of Hecke eigenfunctions. □

For $t_1^{M_1}$ the proof is the same , and for $T_N(\frac{\tilde{a}}{(a_1,a_2)})$ this follows from remark 5.2. To understand that $\mathcal{G}$ is includes all Hecke operators up to linear combinations of them we have the following theorem.

Theorem 5.4. *The group $\mathcal{G} = < T_N(\frac{\tilde{a}}{(a_1,a_2)}), t_2^{\frac{N}{(a_2,N)}}, t_1^{\frac{N}{(a_1,N)}} >$ acts irreducibly over the eigenspaces of $T_N(\tilde{a})$.*

In order to prove Theorem 5.4 we use the following two Lemmas.

Lemma 5.5. *Following the notations so far, we have*

1. $DM_2 \frac{a_2}{D_a} \equiv 0(mod\ N)$ $DM_1 \frac{a_1}{D_a} \equiv 0(mod\ N)$

28

2. $T_N(\frac{\tilde{a}}{D_a})t_i^{M_i} = e_D(k_i)t_i^{M_i}T_N(\frac{\tilde{a}}{D_a})$ $i = 1, 2$ $k_i \in \mathbb{Z}$

3. $(k_1, k_2, D) = 1$

Proof.   1.

$$DM_1\frac{a_2}{D_a} \equiv 0(mod\ N) \iff DM_1a_2 \equiv 0(mod\ ND_a) \iff$$
$$DNa_2 \equiv 0(mod\ ND_aD_2) \iff Da_2 \equiv 0(mod\ D_aD_2) \iff$$
$$a_2 \equiv 0(mod\ \frac{D_a}{D}\frac{D_2}{D})$$

Since $D = (D_a, D_2)$ we get $(\frac{D_a}{D}, \frac{D_2}{D}) = 1$ and so from CRT we get that $a_2 \equiv 0(mod\ \frac{D_a}{D}\frac{D_2}{D}) \iff a_2 \equiv 0(mod\ \frac{D_a}{D} \wedge a_2 \equiv 0(mod\ \frac{D_2}{D}$ which is true and therefore we get that $DM_1\frac{a_2}{D_a} \equiv 0(mod\ N)$. The same holds for a_1.

2. It is clear that due to (4) that $T_N(\frac{\tilde{a}}{D_a})t_i^{M_i} = e_N(\phi)t_i^{M_i}T_N(\frac{\tilde{a}}{D_a})$. what is need to be shown is that $e_N(D\phi) = 1$ which is true due to part (1) of the lemma.

3. Suppose $N = p^\nu$, $a_1 = p^\alpha$, $a_2 = p^\beta$ and therefore $D = p^{\min(\nu,\alpha,\beta)}$, $D_1 = p^{\min(\alpha,\nu)}$, $D_2 = p^{\min(\beta,\nu)}$, $D_a = p^{\min(\alpha,\beta)}$. Denote $j_1 = \min(j|t_1^{jM_1}T_N(\frac{\tilde{a}}{D_a}) = T_N(\frac{\tilde{a}}{D_a})t_1^{jM_1})$ and get $k_1 = \frac{D}{j_1}$, and so is k_2 appropriately. it suffice to show that

$$\max_{i=1,2}(\min(j|t_i^{jM_i}T_N(\frac{\tilde{a}}{D_a}) = T_N(\frac{\tilde{a}}{D_a})t_i^{jM_i})) = D$$

under the assumption that all numbers are prime powers. $M_1\frac{a_1}{D_a} = p^{\nu-\min(\nu,\alpha)+\alpha-\min(\alpha,\beta)}$. If $\nu > \alpha$ then in order to get $jM_1\frac{a_1}{D_a} \equiv 0(mod\ N)$ and therefore $t_1^{jM_1}T_N(\frac{\tilde{a}}{D_a}) = T_N(\frac{\tilde{a}}{D_a})t_1^{jM_1})$, we get $j = p^{\min(\alpha,\beta)} = p^{\min(\alpha,\beta,\nu)} = D$. If $\nu < \alpha, \beta$, $\beta < \alpha$ then for $t_2^{M_2}$ we get $j = p^{\min(\alpha,\beta,\nu)} = D$. All other options are the same.
Suppose now that N, a_1, a_2 are general, then as seen for prime powers, if $p|D$ for some prime either $(k_1, p) = 1$ or $(k_2, p) = 1$ and so we get $(k_1, k_2, D) = 1$

$\square$

Lemma 5.6. *Suppose $\psi(Q)$ is an eigenfunction of $T_N(\frac{\tilde{a}}{D_a})$ with eigenvalue c then $\forall\ m, n$ $t_1^{mM_1}t_2^{nM_2}\psi(Q)$ is also an eigenfunction of $T_N(\frac{\tilde{a}}{D_a})$ with eigenvalue $ce_D(mk_1 + nk_2)$*

Proof. Denote $S = T_N(\frac{\tilde{a}}{D_a}), T_1 = t_1^{M_1}, T_2 = t_2^{M_2}$. Due to (2) we can conclude that

$$ST_1^m T_2^n = e_D(mk_1 + nk_2)T_1^m T_2^n S$$

and therefore, if $S\psi(Q) = c\psi(Q)$ then $ST_1^m T_2^n \psi(Q) = e_D(mk_1 + nk_2)T_1^m T_2^n S\psi(Q) = ce_D(mk_1 + nk_2)T_1^m T_2^n \psi(Q)$ $\qquad\square$

We can now conclude the proof of Theorem (5.4):

Proof. Denote $\mathcal{H}$ to be an eigenspace of U_N. Suppose $0 \neq V \subset \mathcal{H}$ to be a $\mathcal{G}$-invariant subspace. as a $\mathcal{G}$-invariant subspace V is an invariant subspace of $T_N(\frac{\tilde{a}}{D_a})$ and therefore consists an eigenfunction of it denote this as $\psi(Q) \in V$. Also due to $\mathcal{G}$-invariant we can conclude that

$$\forall \ m,n \ T_1^m T_2^n \psi(Q) \in V$$

Now due to Lemma 5.6 $T_1^m T_2^n \psi(Q)$ is an eigenfunction of $T_N(\frac{\tilde{a}}{D_a})$ with eigenvalue $ce_D(mk_1 + nk_2)$ where c is the eigenvalue of $\psi(Q)$.Due to Lemma 3 $(k_1, k_2, D) = 1$ therefore $\#\{e_D(mk_1 + nk_2)|m,n \in \mathbb{Z}\} = D$ and so $ce_D(mk_1 + nk_2)$ are D different numbers leading that V consists D linearly independent eigenfunctions , and since $\mathcal{H}$ is of dimension D (see Lemma 4.2) we get $V = \mathcal{H}$ $\qquad\square$

5.2 Hecke operators for Skew translation

In this section, we follow the procedure done by Marklof and Rudnick in [8]. There it was found that the rate of convergence of the expectation value of the eigenstates is generically bounded by $N^{-1/4+\epsilon}$, and was constructed examples where the rate is arbitrary slow. They also found explicit eigenfunction for them the rate of convergence is always bounded by $N^{-1/2}$. We find here the Hecke operators for the skew translations, and find that the basis of explicit eigenfunctions found in [8], are in fact the unique Hecke basis (basis of joint eigenfunction).

Definition 5.2. *Let α be real number. The following translation*

$$A_\alpha : \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} p + \alpha \\ q + 2p \end{pmatrix} \qquad (24)$$

is called a skew translation.

Remark 5.3. *Notice that A_α can be described as $x \mapsto Ax + \vec{\alpha}$ where $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $\vec{\alpha} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$*

when α is irrational, the skew translation is uniquely ergodic (see proof in [3]). in [8] there are proofs for a quantization of A_α ,its asymptotic behavior and a basis of explicit eigenfunctions as follows:
Define $\mathcal{V}_\alpha$ as $[\mathcal{V}_\alpha \psi](P) = e_N(-(P-a)^2)\psi(P)$ and $\mathcal{F}_N$ be Fourier transform. suppose $\frac{a}{N} \to \alpha$ then

Theorem 5.7.

$$U_N = \mathcal{F}_N \mathcal{V}_\alpha \mathcal{F}_N^{-1} \tag{25}$$

is a quantization of A_α.

Now Denote:

$$D = gcd(a, N) \quad M = \frac{N}{D}$$

Each $1 \le j \le N$ can be described uniquely by a pair (η, l) where

$$\eta \in [1, D] \quad l \in [0, M-1]$$

by $j = \eta + lD$.

Proposition 5.8. *The functions :*

$$\psi_{(\eta,l)}(P) = \begin{cases} \sqrt{D}e_N(-\eta a \nu^2 - \nu l D + a^2 \nu \frac{(M-1)(2M-1)-(\nu-1)(2\nu-1)}{6}) & if\, P \equiv \eta + \nu a \quad mod\ N \\ 0 & if\, P \not\equiv \eta\ mod D \end{cases}$$

are an orthonormal basis of eigenfunctions of $\mathcal{V}_\alpha$ with eigenvalues

$$\phi_{(\eta,l)} = lD - \eta^2 + \eta a - a^2\frac{(M-1)(2M-1)}{6} = lD + c_{a,N}(\eta)$$

Notice that $\psi_{(\eta,l)}$ can be written as $\psi_{(\eta,l)} = \sum_{\nu mod D} c_{j,\nu} \delta_{\eta + \nu a}$ where

$$\delta_a(P) = \begin{cases} 1 & P \equiv a \quad mod\ N \\ 0 & P \not\equiv a \quad mod\ N \end{cases}$$

$m(\phi)$ is bounded by $m(\phi) \ll D^{\frac{1}{2}+\epsilon}$. The rate of convergence of the matrix elements $\langle \text{Op}_N(f)\psi, \psi \rangle$ was found to be bounded by $\frac{1}{\sqrt{M}}$, for a general eigenfunction, which can be arbitrary slow for certain α, and bounded by $N^{-1/4+\epsilon}$, $\forall \epsilon > 0$ for a generic α. For the eigenfunction in Theorem 5.7 the rate of convergence is bounded by $\frac{1}{\sqrt{N}}$, $\forall \alpha$. We see here that the eigenfunctions described are the only Hecke basis of U_N.

Proposition 5.9. *the unitary operator t_2^M satisfy*

$$t_2^M \mathcal{V}_\alpha = \mathcal{V}_\alpha t_2^M$$

Proof. notice that $\mathcal{V}_\alpha = t_1^{-a}\mathcal{V}_0$. clearly $\mathcal{V}_0$ commute with t_2 and all its powers (since they are both multiplying different functions).since $t_1^a t_2^b = e_N(ab)t_2^b t_1^a$ we get that

$$t_1^a t_2^b = t_2^b t_1^a \Leftrightarrow ab \equiv 0(\mod N) \Leftrightarrow b \equiv 0(modM)$$

$\square$

Corollary 5.10. *The group $\mathcal{S} =< t_2^M, \mathcal{V}_\alpha >$ are the Hecke operators of $\mathcal{V}_\alpha$ and therefore its Fourier conjugate $\mathcal{S}^{\mathcal{F}_N}$ are the Hecke operators of U_N.*

Lemma 5.11.   *1. The functions $\psi_{(\eta,l)}$ are eigenfunctions of t_2^M and therefore are eigenfunctions of all $\mathcal{S}$.*

2. $\psi_{(\eta,l)}$ are the only basis of eigenfunctions of all $\mathcal{S}$.

Proof.   1. as seen earlier $\psi_{(\eta,l)}$ can be described as

$$\psi_{(\eta,l)} = \sum_{\nu modD} c_{j,\nu}\delta_{\eta+\nu a}$$

The functions $\delta_{\eta+\nu a}$ are eigenfunctions of t_2^M with eigenvalues $e_N(M\eta)$ and therefore

$$\begin{aligned}
t_2^M(\psi_{(\eta,l)}) &= t_2^M\Big(\sum_{\nu modD} c_{j,\nu}\delta_{\eta+\nu a}\Big) = \\
\sum_{\nu modD} c_{j,\nu}t_2^M(\delta_{\eta+\nu a}) &= \sum_{\nu modD} c_{j,\nu}e_N(M\eta)\delta_{\eta+\nu a} = \\
e_N(M\eta)\sum_{\nu modD} c_{j,\nu}\delta_{\eta+\nu a} &= e_N(M\eta)\psi_{(\eta,l)}
\end{aligned}$$

2. In order to show that this basis is the only basis it suffice to show that there are no two functions with same eigenvalues for both t_2^M and $\mathcal{V}_\alpha$. given two eigenfunctions $\psi_{(\eta,l)}, \psi_{(\eta',l')}$ suppose they have the same eigenvalues in both $t_2^M, \mathcal{V}_\alpha$. The eigenvalues of t_2^M are $e_N(M\eta)$ and so

$$M\eta = M\eta'(mod\ N) \Leftrightarrow \eta \equiv \eta'(modD)$$

since η gets values only modulo D we get $\eta = \eta'$. Now the eigenvalues of $\mathcal{V}_\alpha$ are $lD + c_{a,N}(\eta)$ suppose

$$lD + c_{a,N}(\eta) = l'D + c_{a,N}(\eta')(mod\ N)$$

as we saw $\eta = \eta'$ and therefore $l \equiv l'(mod M)$ and so $l = l'$ and $\psi_{(\eta,l)} = \psi_{(\eta',l')}$.

$\square$

A   Appendices

A.1   Co-prime approximation

Here we show that every $\vec{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ can be approximated by $\frac{(a_1,a_2)}{N}$ such that a_1 is even and $(a_2, N) = 1$ for all N.

Lemma A.1. *Every $\alpha \in \mathbb{R}$ can be approximated by rational numbers $\frac{a}{N}$ such that $(a, N) = 1$ for all N, and $|\alpha - \frac{a}{N}| = O(\frac{1}{N^{1-\epsilon}})$*

Proof. It is known that it is possible to achieve an approximation by rational numbers such that $|\alpha - \frac{a}{N}| = O(\frac{1}{N})$, and so all that is need to be shown is that by inside the interval $[a - c, a + c]$ where c=O(N^ϵ) there exist b such that $(b, N) = 1$. In other words we need to show that for every x there exists $y = O(N^\epsilon)$ such that in the interval $(x, x + y]$ there is a which is invertible modulo N.
Define $\varphi_N(x, y) = \#\{x < m \leqslant x + y | (m, N) = 1\}$.

$$\varphi_N(x, y) = \sum_{x<m\leqslant x+y} \sum_{d|(m,N)} \mu(d) = \sum_{d|N} \mu(d) \sum_{x<m=kd\leqslant x+y} 1 =$$

$$\sum_{d|N} \mu(d) \sum_{\frac{x}{d}<k\leqslant\frac{x+y}{d}} 1 = \sum_{d|N} \mu(d)(\lfloor\frac{y}{d}\rfloor) = y\sum_{d|N} \frac{\mu(d)}{d} + O(\sum_{d|N} 1)$$

Now $O(\sum_{d|N} 1) = O(N^\epsilon)$

Proposition A.2. $\frac{\varphi(N)}{N} \gg \frac{1}{\log N}$

Proof of proposition:

$\frac{\varphi(N)}{N} = \prod_{p|N}(1 - \frac{1}{p})$. Now remember that $\log(1 + x) = x + O(x^2)$ and so

$$\log \prod_{p|N}(1 - \frac{1}{p}) \geq \log \prod_{p \leq N}(1 - \frac{1}{p}) = \sum_{p \leq N} \log(1 - \frac{1}{p}) \ =$$

$$\sum_{p \leq N}\{-\frac{1}{p} + O(\frac{1}{p^2})\} \sim -\log \log N + O(1)$$

therefore $\prod_{p|N}(1 - \frac{1}{p}) \gtrsim e^{-\log \log N + O(1)} = C \frac{1}{\log N}$
which concludes the proposition.

Therefore,

$$\varphi_N(x, y) = y \sum_{d|N} \frac{\mu(d)}{d} + O(\sum_{d|N} 1) = y \frac{\varphi(N)}{N} + O(N^\epsilon) \gtrsim y \frac{1}{\log N} + O(N^\epsilon)$$

so we get that

$$\#\{0 < m | (m, N) = 1, x < m \leqslant x + y\} \gg \frac{y}{\log N} + O(N^\epsilon)$$

so by choosing $y = N^{2\epsilon}$ it is guarantied that there is an invertible a modulo N inside the interval $(x, x + y]$. $\qquad\square$

A.2   Diophantine approximation

Here we give a proof for Theorem 3.5, that states that when $\alpha = (\alpha_1, \ldots, \alpha_l)$ are all algebraic over $\mathbb{Q}$ then α is diophantine. We begin with a little less general Theorem and deduce Theorem 3.5 from it.

Theorem A.3. *Let* $\vec{\alpha} = (\alpha_1, \ldots, \alpha_m)$ *be a basis for a numbers field then for every combination with integer coefficients the following inequality holds*

$$|n_1\alpha_1 + n_m\alpha_m + k| \gg \frac{c(\vec{\alpha})}{\|n\|^{m-1}}$$

Proof. Without loss of generality we can assume that $\alpha_1, \ldots, \alpha_m$ are algebraic integers (since there exits a rational integer a such that $a\alpha_1, \ldots, a\alpha_m$ are algebraic integers).Consider now the norm of $n_1\alpha_1 + \cdots + n_m\alpha_m + k$. since $\vec{\alpha}$ are integers we get that $\mathcal{N}(n_1\alpha_1 + \cdots + n_m + \alpha_m + k) \geqslant 1$,but

$$|\mathcal{N}(n_1\alpha_1 + \cdots + n_m\alpha_m + k)| = |n_1\alpha_1 + \cdots + n_m\alpha_m + k| |\prod_\sigma (n_1\alpha_1 + \cdots + n_m\alpha_m + k)^\sigma|$$

34

and so we get:

$$LHS \leq |n_1\alpha_1 + \cdots + n_m\alpha_m + k|c(\vec{\alpha})\|n\|^{m-1}$$

and so since RHS $\geqslant 1$ we get

$$|n_1\alpha_1 + n_m\alpha_m + k| \gg \frac{c(\vec{\alpha})}{\|n\|^{m-1}}$$

$\square$

Using this Theorem we can prove Theorem 3.5.

A.2.1   Proof of Theorem 3.5

We assume that $\vec{\alpha} = (\alpha_1, \ldots, \alpha_m)$ are linearly independent over $\mathbb{Q}$. $\vec{\alpha}$ can be extended into a basis over $\mathbb{Q}$: $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_m, \ldots, \alpha_D)$ and so for all $n_1, \ldots, n_m, k$ we have:

$$n_1\alpha_1 + \cdots + n_m\alpha_m + k = n_1\alpha_1 + \cdots + n_m\alpha_m + k + 0 \cdot \alpha_{m+1} + \cdots + 0 \cdot \alpha_D \gg$$
$$\frac{c(\vec{\alpha})}{\|\tilde{n}\|^{D-1}} \gg \frac{c(\vec{\alpha})}{\|n\|^{D-1}}$$

and this concludes the proof.

References

[1] M. Degli Esposti, Quantization of the orientation preserving automorphisms of the torus, *Ann. Inst. Poincaré* **58** (1993) 323-341.

[2] M. Degli Esposti, S. Graffi and S. Isola *Classical limit of the quantized hyperbolic toral automorphisms*, Comm. Math Phys. **167** (1995), 471–507.

[3] H. Furstenberg, Strict ergodicity and transformation of the torus, *Amer. J. Math.* **83** (1961) 573-601.

[4] F. Faure, S. Nonnenmacher, S. De Bièvre, Scarred eigenstates for quantum cat maps of minimal periods. *Comm. Math. Phys.* **239** (2003), no. 3, 449–492

[5] J.H. Hannay and M.V. Berry, Quantization of linear maps on a torus — Fresnel diffraction by a periodic grating, *Physica* D **1** (1980) 267-290.

[6] P. Kurlberg and Z. Rudnick, Hecke theory and equidistribution for the quantization of linear maps of the torus, *Duke Math. J.* **103** (2000) 47-77.

[7] J.Marklof , S. O'Keefe Weyl's law and quantum ergodicity for maps with divided phase space, preprint

[8] J. Marklof and Z. Rudnick, Quantum unique ergodicity for parabolic maps, *Geom. Funct. Anal.* **10** (2000) 1554-1578.

[9] W. M. Schnidt, Diophantine approximation Lecture notes in Mathematics, Vol. 785. Springer, Berlin, 1980

[10] S. Zelditch, Uniform distribution of eigenfunctions on compact hyperbolic surfaces, *Duke Math. J.* **55** (1987) 919-941.